SECTION 2.6: CONTINUITY

RECALL: A function f is said to be **continuous** at real number a if $\lim_{x \to a} f(x) = f(a)$.

Computationally, if f is continuous at $x = a$, then to find $\lim_{x \to a} f(x)$, we can just evaluate $f(a)$.

The equation $\lim_{x \to a} f(x) = f(a)$ is really saying (at least) three different things:

Continuity Checklist: f is continuous at $x = a$ means:

- $f(a)$ exists.
- $\lim_{x \to a} f(x)$ exists.
- $\lim_{x \to a} f(x) = f(a)$

EXAMPLE 1: Explain why the following functions are not continuous at the indicated value.

1. $f(x) = \dfrac{3x}{(x-2)^2}$ at $x = 2$. Ans: Neither $f(2)$ nor $\lim_{x \to 2} f(x)$ exist.

2. $g(x) = \dfrac{\sin(x)}{x}$ at $x = 0$. Ans: $g(0)$ does not exist.

3. $h(x) = \begin{cases} 3x & \text{if } x < 1 \\ 2x^2 & \text{if } x \geq 1 \end{cases}$ at $x = 1$. Ans: $\lim_{x \to 0} h(x)$ does not exist.

EXAMPLE 2: Let $F(x) = \begin{cases} x^2 - 3x + 1 & \text{if } x \leq 4 \\ cx + 7 & \text{if } x > 4 \end{cases}$. Find the value of the constant c so f is continuous at $x = 4$.

In order for F to be continuous at $x = 4$, $F(4)$ must exist, $\lim_{x \to 4} F(x)$ must exist, and $\lim_{x \to 4} F(x) = F(4)$.

We have that $F(4) = (4)^2 - 3(4) + 1 = 5$ exists. We now analyze $\lim_{x \to 4} F(x)$.

Since F is a piecewise-defined function whose definition changes at $x = 4$, we check $\lim_{x \to 4^-} F(x)$ and $\lim_{x \to 4^+} F(x)$.

We have $\lim_{x \to 4^-} F(x) = \lim_{x \to 4^-} \left(x^2 - 3x + 1\right) = (4)^2 - 3(4) + 1 = 5$ and $\lim_{x \to 4^+} F(x) = \lim_{x \to 4^-} (cx + 7) = 4c + 7$.

In order for $\lim_{x \to 4} F(x)$ to exist, we need $\lim_{x \to 4^-} F(x) = \lim_{x \to 4^+} F(x)$ which means $4c + 7 = 5$ or $c = -\dfrac{1}{2}$.

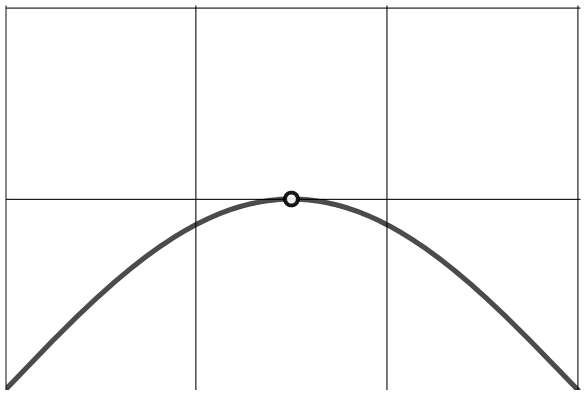
If $c = -\dfrac{1}{2}$, then $\lim_{x \to 4^-} F(x) = \lim_{x \to 4^+} F(x) = 5$ so $\lim_{x \to 4} F(x)$ exists.

Moreover, $\lim_{x \to 4} F(x) = 5 = F(4)$ so F is continuous at $x = 4$.

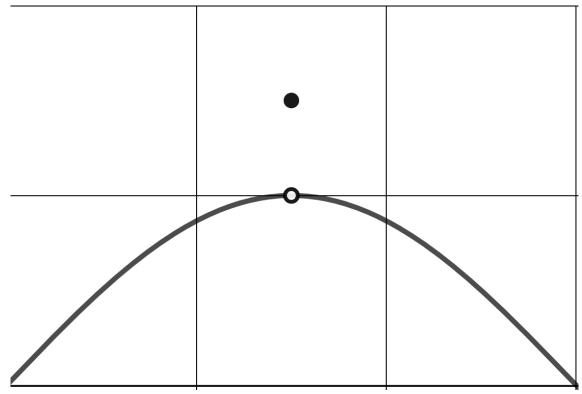
TYPES OF DISCONTINUITY: Suppose f is not continuous at $x = a$. Then we say:

- f has a **removable discontinuity** at $x = a$ if $\lim_{x \to a} f(x)$ exists.

  NOTE: Graphically we could have . . .



$f(a)$ is undefined.

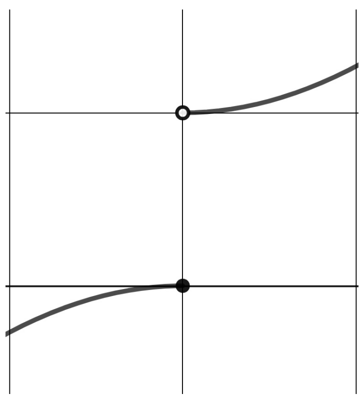


$\lim_{x \to a} f(x) \neq f(a)$

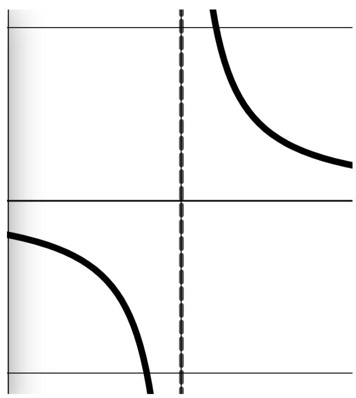
  We can **remove** removable discontinuities by defining the function $f(a)$ to be $\lim_{x \to a} f(x)$.

- f has a **non-removable (or essential) discontinuity** at $x = a$ if $\lim_{x \to a} f(x)$ does not exist.

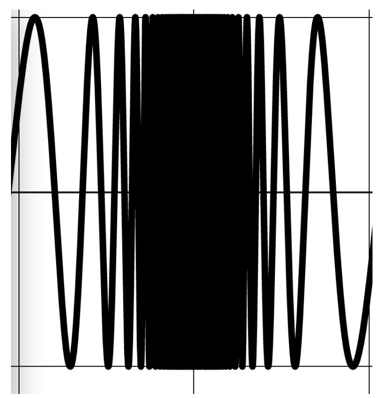
  NOTE: Graphically, we could have . . .



'Jump' Discontinuity



'Infinite' Discontinuity



'Oscillation' Discontinuity

  There are no 'quick fixes' with non-removable discontinuities.

EXAMPLE 3:

1. $f(x) = \dfrac{3x}{(x-2)^2}$ has a non-removable infinite discontinuity at $x = 2$ since $\lim_{x \to 2} \dfrac{3x}{(x-2)^2} = \infty$.

2. $g(x) = \dfrac{\sin(x)}{x}$ has a removable discontinuity at $x = 0$ since $g(0)$ doesn't exist, but $\lim_{x \to 0} \dfrac{\sin(x)}{x} = 1$.

   We can **remove** the discontinuity at $x = 0$ by **redefining** the function g as $g(x) = \begin{cases} \dfrac{\sin(x)}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

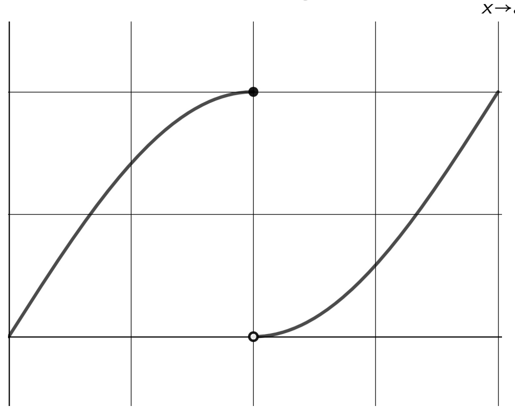
3. $h(x) = \begin{cases} 3x & \text{if } x < 1 \\ 2x^2 & \text{if } x \geq 1 \end{cases}$ has a jump discontinuity at $x = 1$ since $\lim_{x \to 1^-} h(x) = 3$ but $\lim_{x \to 1^+} h(x) = 2$.

Let's take a look at $h(x) = \begin{cases} 3x & \text{if } x < 1 \\ 2x^2 & \text{if } x \geq 1 \end{cases}$. We know $\lim_{x \to 1} h(x)$ does not exist so h is not continuous at $x = 1$.

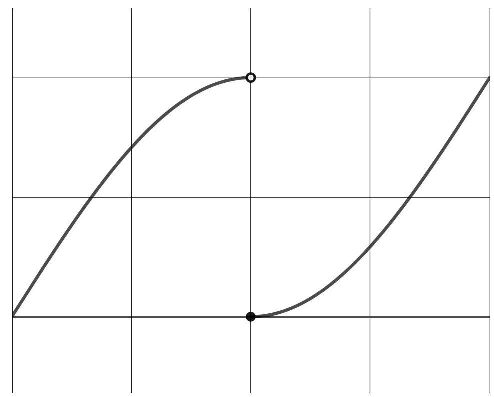
However, since $\lim_{x \to 1^+} h(x) = 2 = h(1)$, we say h is continuous **from the right** at $x = 1$.

ONE-SIDED CONTINUITY: A function is said to be:

- **continuous from the left** at $x = a$ if $\lim_{x \to a^-} f(x) = f(a)$.
- **continuous from the right** at $x = a$ if $\lim_{x \to a^+} f(x) = f(a)$.



continuous from the left



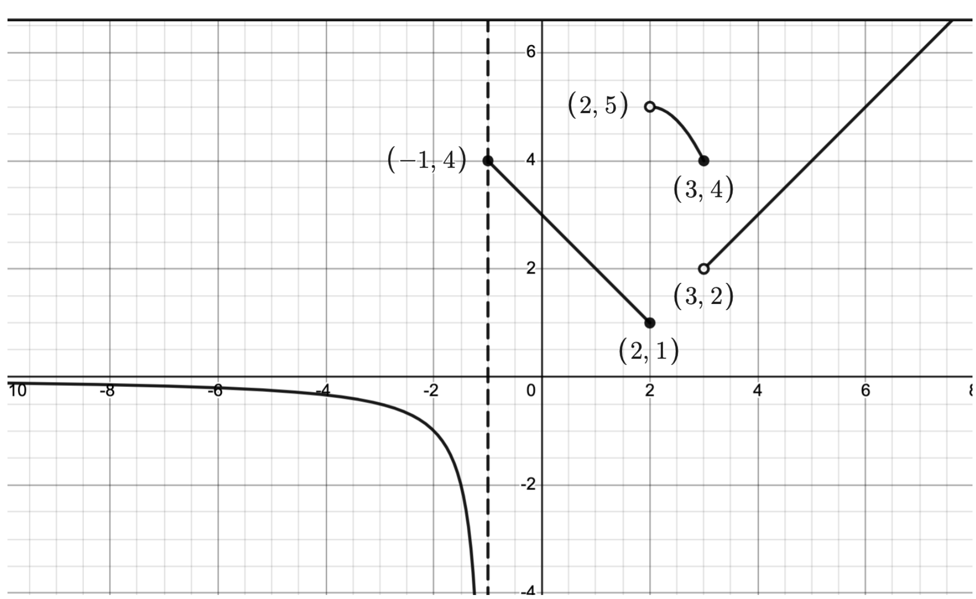
continuous from the right

It follows that a function is continuous at $x = a$ if it is continuous from both the left and right at $x = a$.

INTERVALS OF CONTINUITY: A function is said to be continuous

- on the open interval (a, b) if f is continuous at every point in the interval (a, b).
- on the interval $[a, b)$ if f is continuous on (a, b) and continuous **from the right** at $x = a$.
- on the interval $(a, b]$ if f is continuous on (a, b) and continuous **from the left** at $x = b$.
- on the interval $[a, b]$ if f is continuous on $[a, b)$ and $(a, b]$. That is: f is continuous on (a, b), f is continuous from the right at $x = a$ and f is continuous from the left at $x = b$.

EXAMPLE 4: List the intervals of continuity for the following function. What assumptions are you making?



Ans: $(-\infty, -1)$, $[-1, 2]$, $(2, 3]$, $(3, \infty)$

Retuning to the functions we've studied in **EXAMPLE 1** and **EXAMPLE 3**, we have:

1. $f(x) = \dfrac{3x}{(x-2)^2}$ is continuous on $(-\infty, 2)$ and again on $(2, \infty)$.

2. $g(x) = \dfrac{\sin(x)}{x}$ is continuous on $(-\infty, 0)$ and again on $(0, \infty)$.

3. $h(x) = \begin{cases} 3x & \text{if } x < 1 \\ 2x^2 & \text{if } x \geq 1 \end{cases}$ is continuous on $(-\infty, 1)$ and again on $[1, \infty)$.

NOTE: We do not use the union '$\cup$' symbol when listing intervals of continuity. To see this, we refer to the function h in the last example. As sets of real numbers, $(-\infty, 1) \cup [1, \infty)$ is the same as $(-\infty, \infty)$, but h is not continuous on $(-\infty, \infty)$. The parenthesis on the 1 in '$(-\infty, 1)$' and bracket on the 1 in interval '$[1, \infty)$' carry specific information about h being only continuous from the right at $x = 1$. We lose this information if we combine these intervals using the usual set notation.

EXAMPLE 5 (VIDEO): Classify the discontinuities of the given function and list the interval(s) of continuity.

If a discontinuity is removable, (re-)define the function so as to remove the discontinuity.

1. $f(x) = \dfrac{x^2 - x - 6}{x^2 - 9}$

Ans: Non-removable VA: $x = -3$; removable hole: $\left(3, \dfrac{5}{6}\right)$. I. o. C.: $(-\infty, -3)$, $(-3, 3)$, $(3, \infty)$.

If we define $f(3) = \dfrac{5}{6}$, this new function has I. o. C. of $(-\infty, -3)$, $(-3, \infty)$.

2. $g(x) = \dfrac{|x^2 - 25|}{x - 5}$

Ans: Non-removable jump at $x = 5$; I. o. C.: $(-\infty, 5)$, $(5, \infty)$.

3. $h(x) = \sqrt{16 - x^2}$

Ans: I. o. C.: $[-4, 4]$.

EXAMPLE 6 (VIDEO): Graph a function f satisfying all of the following criteria:

- the intervals of continuity for f are $(-\infty, 3]$ and $(3, \infty)$
- $\lim\limits_{x \to 3^-} f(x) = 2$ and $\lim\limits_{x \to 3^+} f(x) = 5$

COMBINATIONS OF CONTINUOUS FUNCTIONS ARE CONTINUOUS:

Suppose f and g are continuous at $x = a$. Then:

- $f + g$ and $f - g$ are continuous at $x = a$.
- $f \cdot g$ and $\dfrac{f}{g}$ are continuous at $x = a$ (provided $g(a) \neq 0$.)
- if h is continuous at $f(a)$, then $h \circ f$ is continuous at $x = a$.

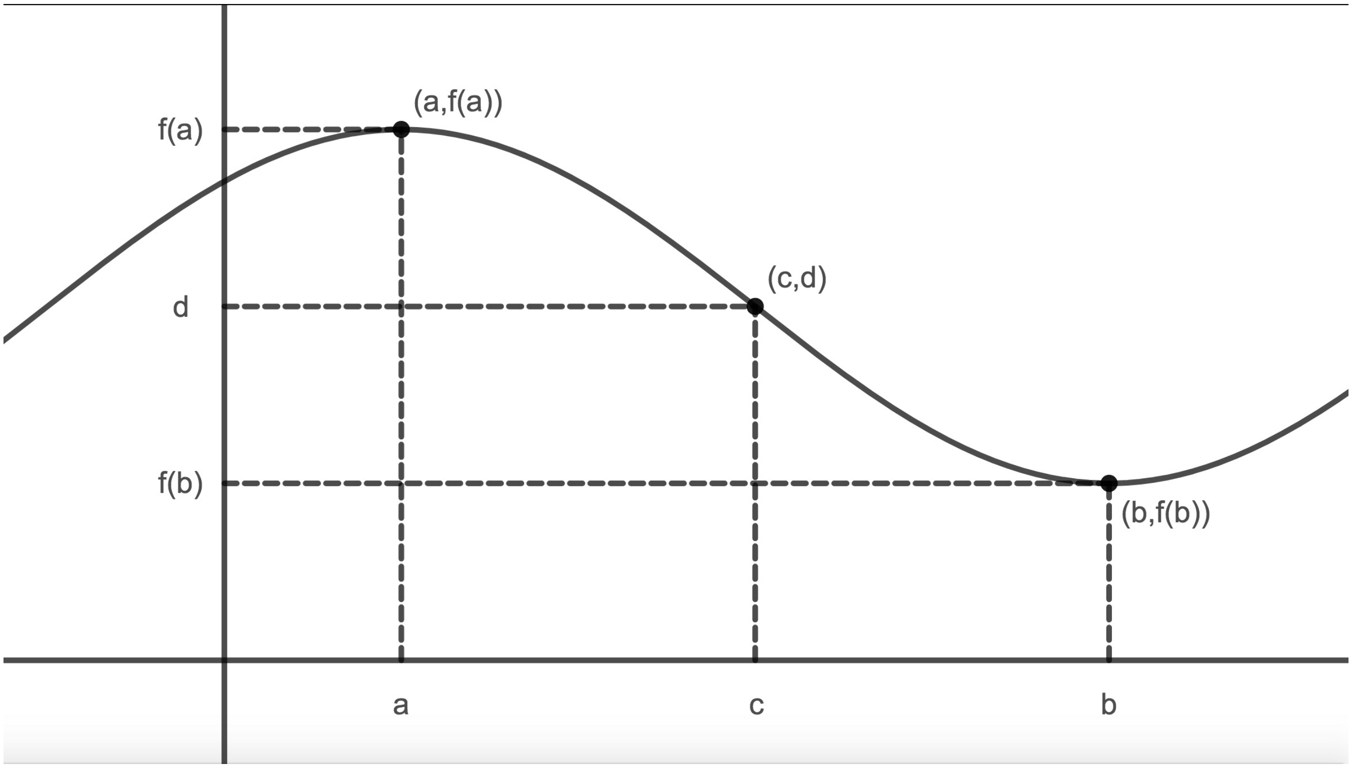
The sum, difference, product, and quotient laws above are direct consequences of the corresponding limit laws.

The final statement about composition, while intuitively true, is a bit more involved.

One of the most important properties of continuous functions is the following:

THE INTERMEDIATE VALUE THEOREM (IVT) (VIDEO):

If f is continuous on an interval containing real numbers a and b and d is any real number between $f(a)$ and $f(b)$, then there is at least one real number c between a and b with $f(c) = d$.



NOTE: The number 'd' above is an 'intermediate' output value since it lives between the outputs $f(a)$ and $f(b)$ while the value 'c' is an 'intermediate' input value since it lives between the inputs a and b. The IVT says for continuous functions defined over an interval, every intermediate output value comes from an intermediate input value. In other words, continuous functions map intervals onto intervals.

EXAMPLE 7 (VIDEO): Prove the function $f(x) = x - \cos(x)$ has a zero between $x = 0$ and $x = \dfrac{\pi}{2}$.

Recall a 'zero' of f is a solution to $f(x) = 0$. Hence, the number '0' is our intermediate output value (or 'd' from the IVT.) We note that $f(0) = -1 < 0$ and $f\left(\dfrac{\pi}{2}\right) = \dfrac{\pi}{2} > 0$ so $f(0) < d < f\left(\dfrac{\pi}{2}\right)$. Since f is a difference of continuous functions (x and $\cos(x)$), the IVT guarantees at least one number c between 0 and $\frac{\pi}{2}$ with $f(c) = 0$.

CHALLENGE: Assuming temperature is a continuous function, show that at any given time, there are two antipodal points on the earth with the same temperature.

HOMEWORK: Section 2.6: 1 - 73 every other odd.